A method for computing waveguide scattering matrices in the presence of discrete spectrum

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Abstract

A waveguide G lies in $\mathbb{R}^{n+1}$, $n \geq 1$, and outside a large ball coincides with the union of finitely many non-overlapping semi-cylinders ("cylindrical ends"). The waveguide is described by the operator $\{\mathcal{L}(x, D_x) - \mu, \mathcal{B}(x, D_x)\}$ of an elliptic boundary value problem in G, where $\mathcal{L}$ is a matrix differential operator, $\mathcal{B}$ is a boundary operator, and μ is a spectral parameter. The operator $\{\mathcal{L}, \mathcal{B}\}$ is self-adjoint with respect to a Green formula. The role of $\mathcal{L}$ can be played, e.g., by the Helmholtz operator, by the operators in elasticity theory and hydrodynamics. As approximation for a row of the scattering matrix $S(\mu)$, we take the minimizer of a quadratic functional $J^R(\cdot, \mu)$. To construct the functional, we solve an auxiliary boundary value problem in the bounded domain obtained by truncating the cylindrical ends of the waveguide at distance R. As $R \to \infty$, the minimizer $a(R, \mu)$ tends with exponential rate to the corresponding row of the scattering matrix uniformly on every finite closed interval of the continuous spectrum not containing the thresholds. Such an interval may contain eigenvalues of the waveguide with eigenfunctions exponentially decaying at infinity ("trapped modes"). Eigenvalues of this sort, as a rule, occur in waveguides of complicated geometry. Therefore, in applications, the possibility to avoid worrying about (probably not detected) trapped modes turns out to be an important advantage of the method.

1 Introduction

The waveguide to be considered in the paper occupies a domain G in $\mathbb{R}^{n+1}$ having several cylindrical outlets to infinity ("cylindrical ends"). This means that outside a large ball centered at the origin the domain G coincides with the union of non-overlapping semicylinders $\Pi_+^1, \dots, \Pi_+^P$; here $\Pi_+^p = \{(y^p, t^p) : y^p \in \Omega^p, t^p > 0\}$, (y^p, t^p) are local coordinates in Π_+^p, and the cross-section Ω^p of the cylinder Π^p is a bounded n-dimensional domain with smooth boundary $\partial\Omega^p$. The waveguide is described by the operator $\{\mathcal{L}(x, D_x) - \mu, \mathcal{B}(x, D_x)\}$ of an elliptic boundary value problem in G, where $\mathcal{L}$ is a matrix differential operator, $\mathcal{B}$ is a boundary condition operator, and μ is a spectral parameter. The operator $\{\mathcal{L}, \mathcal{B}\}$ is self-adjoint with respect to a Green formula. At infinity, the coefficients of $\mathcal{L}$ and $\mathcal{B}$ stabilize with exponential rate in every semicylinder Π_+^p to functions independent of the axial variable.

Let us consider the homogeneous problem

$$\begin{aligned} (\mathcal{L}(x, D_x) - \mu)u(x) = 0, \qquad x \in G, \\ \mathcal{B}(x, D_x)u(x) = 0, \qquad x \in \partial G. \end{aligned} \tag{1.1}$$

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We assume that the interval $[\mu_1, \mu_2] \subset \mathbb{R}$ belongs to the continuous spectrum of the operator $\{\mathcal{L} - \mu, \mathcal{B}\}$ and contains no threshold values of the spectral parameter. In other words, for every $\mu \in [\mu_1, \mu_2]$ there exists the same (finite) number of solutions to the problem (1.1) linearly independent modulo $L_2(G)$; such solutions are called eigenfunctions of the continuous spectrum. The interval $[\mu_1, \mu_2]$ may contain eigenvalues of the problem (1.1) with eigenfunctions in $L_2(G)$. Any eigenfunction in $L_2(G)$ is exponentially decaying at infinity while the eigenvalues are of finite multiplicity and can not accumulate in $[\mu_1, \mu_2]$. Thus, when $\mu \in [\mu_1, \mu_2]$ turns out to be an eigenvalue, the number of bounded solutions linearly independent in the ordinary sense increases and yet the number of solutions linearly independent modulo $L_2(G)$ (or, equivalently, modulo exponentially decaying terms) remains constant on $[\mu_1, \mu_2]$; denote the number by M. For any $\mu \in [\mu_1, \mu_2]$ in the space of continuous spectrum eigenfunctions there exists a basis $Y_1(\cdot, \mu), \dots, Y_M(\cdot, \mu)$ modulo $L_2(G)$ such that

$$Y_j(x, \mu) = u_j^+(x, \mu) + \sum_{k=1}^{M} S_{jk}(\mu) u_k^-(x, \mu) + O(e^{-\varepsilon|x|})$$

for $|x| \to \infty$ and $j = 1, \dots, M$; here ε is a sufficiently small positive number, $u_j^+(\cdot, \mu)$ are incoming waves while $u_j^-(\cdot, \mu)$ are outgoing ones (precise definitions see in 2.2). The matrix $S(\mu) = \|S_{jk}(\mu)\|$ is unitary; it is called the scattering matrix.

The paper is devoted to justification of an approximate computation method for the scattering matrix. A detailed description of the method has been given in 2.3. In brief as an approximation to the l-th row $S_l(\mu) = (S_{l,1}(\mu), \dots, S_{l,M}(\mu))$ of the scattering matrix we take the minimizer $a(R, \mu)$ of a quadratic functional $J_l^R(\cdot, \mu)$. The functional is constructed by solving an auxiliary boundary value problem in the bounded domain G^R obtained from G by cutting off the cylindrical ends at a sufficiently large distance R from the origin. In the present paper, we prove that for $R \geqslant R_0$ and all $\mu \in [\mu_1, \mu_2]$ there holds the estimate

$$\|a(R, \mu) - S_l(\mu)\| \leqslant C e^{-\Lambda R} \tag{1.2}$$

with some positive numbers Λ and C that are independent of R and μ. Thus, as $R \to \infty$, the minimizer $a(R, \mu)$ tends to the corresponding row of the scattering matrix with exponential rate uniformly with respect to $\mu \in [\mu_1, \mu_2]$.

As mentioned above, the interval $[\mu_1, \mu_2]$ of the continuous spectrum may contain eigenvalues of the operator $\{\mathcal{L} - \mu, \mathcal{B}\}$. In nonhomogeneous waveguides of complicated geometry, as a rule, there occur trapped modes, that is, eigenfunctions exponentially decaying at infinity. Therefore, in applications, the possibility to avoid worrying about (probably not detected) trapped modes turns out to be an important advantage of the method.

For the Helmholtz operator in a close situation, the method under discussion was suggested in [1]. The justification of the method in [1] made use of Proposition 3 there (presented without proof), which is valid only if the interval $[\mu_1, \mu_2]$ is free from the eigenvalues of the waveguide; this restriction was not indicated in Proposition 3. (In our context the restriction would mean that $[\mu_1, \mu_2]$ must be free from the eigenvalues of the operator $\{\mathcal{L} - \mu, \mathcal{B}\}$ in G.) In the presence of waveguide eigenvalues in $[\mu_1, \mu_2]$, the inequality (1.2) was proved for the Helmholtz operator in [2]. The proof given in the present paper for the general elliptic problems is new for the Helmholtz operator as well and is simpler than that in [2]. In contrast to [2], we here do not construct an approximate solution to the problem

in G^R. Instead, we use a simple estimate on its solutions which is uniform with respect to $R \geq R_0$ and $\mu \in [\mu_1, \mu_2]$ and is indifferent to the presence in $[\mu_1, \mu_2]$ of eigenvalues of the operator $\{\mathcal{L} - \mu, \mathcal{B}\}$ in G.

The statement of basic boundary value problem, the precise definition of scattering matrix, the detailed formulation of computational method, and the statement of principal Theorem 2.3 are given in §2. The solvability of an auxiliary problem in the domain G^R is discussed in §3. In §4, we complete the justification of method by proving Theorem 2.3.

2 Statement of the problem. Formulation of the results

2.1 Boundary value problem

Let G be a domain in $\mathbb{R}^{n+1}$ coinciding, outside a large ball, with the union $\Pi_+^1 \cup \ldots \cup \Pi_+^P$ of non-overlapping semicylinders $\Pi_+^p = \{(y^p, t^p) : y^p \in \Omega^p, t^p > 0\}$, where (y^p, t^p) are local coordinates in Π_+^p, the cross-section Ω^p of Π_+^p being a bounded domain in $\mathbb{R}^n$. The boundary ∂G of G is supposed to be smooth. We introduce a formally self-adjoint $(k \times k)$-matrix $\|\mathcal{L}_{ij}(x, D_x)\|$ of differential operators in G, where $\operatorname{ord} \mathcal{L}_{ij} = \tau_i + \tau_j$ with nonnegative integers τ_j and $\tau_1 + \ldots + \tau_k = m$. We also assume that the Green formula

$$(\mathcal{L}u, v)_G + (\mathcal{B}u, \mathcal{Q}v)_{\partial G} = (u, \mathcal{L}v)_G + (\mathcal{Q}u, \mathcal{B}v)_{\partial G} \tag{2.1}$$

holds for all $u, v \in C_c^\infty(\overline{G})$, while the $(m \times k)$-matrices $\mathcal{B} = \|\mathcal{B}_{qj}\|$ and $\mathcal{Q} = \|\mathcal{Q}_{qj}\|$ consist of differential operators such that $\operatorname{ord} \mathcal{B}_{qj} = \sigma_q + \tau_j$, σ_q being an integer, and $\operatorname{ord} \mathcal{B}_{qj} + \operatorname{ord} \mathcal{Q}_{qi} \leqslant \tau_j + \tau_i - 1$; here $(\cdot, \cdot)_G$ and $(\cdot, \cdot)_{\partial G}$ stand for the inner products on $L_2(G)$ and $L_2(\partial G)$. The coefficients of $\mathcal{L}$, $\mathcal{B}$, and $\mathcal{Q}$ are smooth in $\overline{G}$. We suppose that the operator $\{\mathcal{L}, \mathcal{B}\}$ of boundary value problem in G is elliptic.

Remark 2.1. We have used the (widest) ellipticity definition in the sense of Agmon-Douglis-Nirenberg in order to include in consideration, among others, some hydrodynamics problems. The other ellipticity definitions can be obtained from that as special cases. For instance, the scalar case corresponds to $k = 1$, $\tau_1 = m$, and $\operatorname{ord}\mathcal{L} = 2m$; the boundary operator $\mathcal{B}$ is a column $\{\mathcal{B}_1, \ldots, \mathcal{B}_m\}$ with $\operatorname{ord}\mathcal{B}_h = m_h$. A detailed description of various ellipticity definitions as well as examples can be found, e.g., in [3].

Let us describe the coefficients of $\mathcal{L}$ and $\mathcal{B}$ in a neighborhood of infinity. Denote by $\{L^p, B^p\}$ an elliptic boundary value problem operator in the cylinder $\Pi^p = \Omega^p \times \mathbb{R}$ with coefficients independent of $t^p \in \mathbb{R}$ and smooth in $\overline{\Omega^p}$, while $\operatorname{ord} L_{ij}^p = \operatorname{ord} \mathcal{L}_{ij}$ and $\operatorname{ord} B_{ij}^p = \operatorname{ord} \mathcal{B}_{ij}$. We assume that any coefficient a of $\{\mathcal{L}, \mathcal{B}\}$ satisfies

$$D^\alpha \left(a(y^p, t^p) - a^p(y^p)\right) = O(\exp(-\delta t^p)) \tag{2.2}$$

in Π_+^p for $t^p \to +\infty$, where a^p is the corresponding coefficient of $\{L^p, B^p\}$ and δ is a positive number. From (2.1) it follows that there holds the Green formula in every cylinder Π^p obtained from (2.1) by changing G for Π^p and $\mathcal{L}$, $\mathcal{B}$, and $\mathcal{Q}$ for L^p, B^p, and Q^p.

We consider the boundary value problem

$$\begin{cases} \mathcal{L}(x, D_x)u(x) - \mu u(x) = 0, & x \in G, \\ \mathcal{B}(x, D_x)u(x) = 0, & x \in \partial G, \end{cases} \tag{2.3}$$

with spectral parameter $\mu \in \mathbb{C}$. A number μ is called an eigenvalue of the operator $\{\mathcal{L}, \mathcal{B}\}$, if there exists a nonzero function $u \in L_2(G)$ smooth in $\overline{G}$ and satisfying (2.3). Such a function is called an eigenfunction corresponding to the eigenvalue μ. From the Green formula it follows that any eigenvalue is real. Every eigenfunction admits the estimate $u(y^p, t^p) = O(\exp(-\varkappa t^p))$ for $t^p \to +\infty$ in each Π_+^p with a certain $\varkappa > 0$. For any eigenvalue there exist at most finitely many linearly independent eigenfunctions.

2.2 Space of waves. Scattering matrix

To simplify notation, we sometimes drop the superscript p, if the context excludes misunderstanding. In every domain $\Omega = \Omega^p$ we introduce the operator pencil

$$\mathbb{C} \ni \lambda \mapsto \mathfrak{A}(\lambda, \mu) = \{L(\lambda) - \mu I, B(\lambda)\},$$

where $L(\lambda) = L(y, D_y, \lambda)$, $B(\lambda) = B(y, D_y, \lambda)$, so we have the pencils $\mathfrak{A}^1(\lambda, \mu)$, …, $\mathfrak{A}^P(\lambda, \mu)$. Let us fix, for the time being, the parameter μ. Considering λ as spectral parameter of the pencils, we shall use the same terminology as in [4]. The spectrum of $\mathfrak{A}(\cdot, \mu)$ is symmetric about the real axis and consists of normal eigenvalues, that is, isolated eigenvalues of finite algebraic multiplicity. Every strip $\{\lambda \in \mathbb{C} : |\operatorname{Im} \lambda| < h < \infty\}$ contains at most finitely many eigenvalues. It is known (see [5]) that the total algebraic multiplicity of the eigenvalues of $\mathfrak{A}(\cdot, \mu)$ in the strip $\{\lambda \in \mathbb{C} : |\operatorname{Im} \lambda| < h\}$ is even for any $h > 0$; in particular, the total algebraic multiplicity of the real eigenvalues is even.

If μ is not a threshold, then for any real eigenvalue of $\mathfrak{A}^p(\cdot, \mu)$ there exist only eigenvectors and no generalized eigenvectors. We number all real eigenvalues of the pencil $\mathfrak{A}^p(\cdot, \mu)$ counted according to their (geometric) multiplicity. Let $\lambda_1^p, \dots, \lambda_{2M^p}^p$ be all such eigenvalues and let $\varphi_1^p, \dots, \varphi_{2M^p}^p$ be the corresponding eigenvectors. The functions

$$\Pi^p \ni (y, t) \mapsto u_k^p(y, t) = \exp(i\lambda_k^p t)\varphi_k^p(y)$$

satisfy the homogeneous problem

$$\begin{cases} (L^p(y, D_y, D_t) - \mu)v(y, t) = 0, & (y, t) \in \Pi^p; \\ B^p(y, D_y, D_t)v(y, t) = 0, & (y, t) \in \partial\Pi^p. \end{cases} \tag{2.4}$$

Introduce

$$q^p(u, v) := (L^p u, v)_{\Pi^p} + (B^p u, Q^p v)_{\partial\Pi^p} - (u, L^p v)_{\Pi^p} - (Q^p u, B^p v)_{\partial\Pi^p}.$$

If $u, v \in C_c^\infty(\overline{\Pi^p})$, then $q^p(u, v) = 0$. Assume that $\chi \in C^\infty(\mathbb{R})$, $\chi(t) = 1$ for $t \geqslant 2$ and $\chi(t) = 0$ for $t \leqslant 1$. The form q^p extends to the pairs $\{\chi u_j^p, \chi u_k^p\}$, and there is valid the following assertion (see [5]).

Proposition 2.2. *The eigenvectors $\{\varphi_k^p\}$ can be chosen to satisfy $q^p(\chi u_j^p, \chi u_k^p) = \pm i\delta_{jk}$, while the sign is determined and cannot be taken arbitrarily.*

We extend the functions χu_k by zero from Π^p_+ to the domain G keeping the same notation for the extended functions. Introduce a space E of $w \in C^\infty(\overline{G})$ satisfying $D^\alpha w(x) = O(\exp(-\beta|x|))$ as $|x| \to +\infty$ for any multiindex α and some positive $\beta < \delta$, where δ is the number in (2.2). Let $\mathcal{W}$ stand for the linear span of functions $\chi u_k + w$ with $w \in E$. The form

$$q(u,v) = (\mathcal{L}u, v)_G + (\mathcal{B}u, \mathcal{Q}v)_{\partial G} - (u, \mathcal{L}v)_G - (\mathcal{Q}u, \mathcal{B}v)_{\partial G} \tag{2.5}$$

takes finite values for any $u, v \in \mathcal{W}$. It is evident that $q(u,v) = -\overline{q(v,u)}$ and $q(u,u) \in i\mathbb{R}$. By definition, $u \in \mathcal{W}$ is an incoming (outgoing) wave, if $iq(u,u) > 0$ ($iq(u,u) < 0$). In the space $\mathcal{W}$ one can choose the basis

$$u_1^+, \dots, u_M^+, u_1^-, \dots, u_M^-, \qquad M = \sum_{p=1}^{P} M^p, \tag{2.6}$$

subject to the conditions

$$q(u_j^\pm, u_k^\pm) = \mp i\delta_{jk}, \quad q(u_j^\pm, u_k^\mp) = 0, \quad j,k = 1, \dots, M; \tag{2.7}$$

here $u_1^+, \dots, u_M^+$ are incoming waves and $u_1^-, \dots, u_M^-$ are outgoing ones.

Let γ be a positive number such that $\gamma < \delta$ and the strip $\{\lambda \in \mathbb{C} : |\operatorname{Im}\lambda| \le \gamma)\}$ contains only real eigenvalues of the pencils $\mathfrak{A}^p(\cdot, \mu)$, $p = 1, \dots, P$, for all $\mu \in [\mu_1, \mu_2]$. It is known [5] that in the space of bounded solutions to the problem (2.3) there exist elements $Y_1(\cdot, \mu) \dots, Y_M(\cdot, \mu)$ such that

$$Y_j(x, \mu) = u_j^+(x, \mu) + \sum_{k=1}^{M} S_{jk}(\mu) u_k^-(x, \mu) + O(e^{-\gamma|x|}) \tag{2.8}$$

as $|x| \to \infty$. If the number μ is not an eigenvalue of the problem (2.3), then $Y_j(\cdot, \mu)$ are uniquely determined and form a basis in the space of bounded solutions to the homogeneous problem (2.3), that is, in the space of continuous spectrum eigenfunctions. Otherwise, any $Y_j(\cdot, \mu)$ is determined up to an eigenfunction of (2.3) belonging to $L_2(G)$. Then any bounded solution of (2.3) can be represented by a linear combination of $Y_j(\cdot, \mu)$ up to an eigenfunction in $L_2(G)$.

The matrix $S(\mu) = \|S_{jk}(\mu)\|_{j,k=1}^M$ in (2.8) has uniquely been determined for all $\mu \in [\mu_1, \mu_2]$; it is independent of the arbitrariness in the definition of $Y_j(\cdot, \mu)$ when μ is an eigenvalue. The matrix $S(\mu)$ is called the scattering matrix. It is unitary for all μ.

In what follows we do not usually indicate the dependence of $u_j^\pm$, Y_j, etc., on the spectral parameter μ. The context excludes misunderstanding.

2.3 Method for computing the scattering matrix. Formulation of the principal theorem

For large R we introduce

$$\Pi_+^{p,R} = \{(y^p, t^p) \in \Pi^p : t^p > R\}, \quad G^R = G \setminus \cup_{p=1}^N \Pi_+^{p,R}$$

and set

$$\partial G^R \setminus \partial G = \Gamma^R = \cup_p \Gamma^{p,R}, \quad \Gamma^{p,R} = \{(y^p, t^p) \in \Pi^p : t^p = R\}.$$

Then

$$(\mathcal{L}u, v)_{G^R} + (\mathcal{B}u, \mathcal{Q}v)_{\partial G^R\setminus\Gamma^R} + (\mathcal{N}u, \mathcal{D}v)_{\Gamma^R} = (u, \mathcal{L}v)_{G^R} + (\mathcal{Q}u, \mathcal{B}v)_{\partial G^R\setminus\Gamma^R} + (\mathcal{D}u, \mathcal{N}v)_{\Gamma^R}, \quad (2.9)$$

where $\mathcal{D}$ and $\mathcal{N}$ are $(m \times k)$–matrices of differential operators, $\mathcal{D}$ being a Dirichlet system (see [8]). An example of Dirichlet system is presented by the matrix consisting of m rows of the form $e^{(j)}\partial_\nu^h$, where $j = 1, \dots, k$, $h = 1, \dots, \tau_j - 1$, $e^{(j)} = (\delta_{1,j}, \dots, \delta_{k,j})$, and ν is the outward normal to Γ^R.

We look for the row $(S_{l1}, \dots, S_{lM})$ of the scattering matrix $S = S(\mu)$. As approximation to the row, we take the minimizer of a quadratic functional. To construct such a functional we consider the problem

$$\begin{aligned} (\mathcal{L}(x, D_x) - \mu)\mathcal{X}_l^R &= 0, \quad x \in G^R, \\ \mathcal{B}(x, D_x)\mathcal{X}_l^R &= 0, \quad x \in \partial G^R \setminus \Gamma^R, \\ (\mathcal{N} + i\zeta\mathcal{D})\mathcal{X}_l^R &= (\mathcal{N} + i\zeta\mathcal{D})(u_l^+ + \textstyle\sum_{j=1}^M a_j u_j^-), \ x \in \Gamma^R, \end{aligned} \quad (2.10)$$

where ζ is a fixed number in $\mathbb{R} \setminus \{0\}$ and $a_1, \dots, a_M$ are complex numbers.

Let us explain the origin of the problem. A solution Y_l of the homogeneous problem (2.3) satisfies the two first equations (2.10). The asymptotics (2.8) can be differentiated so

$$(\mathcal{N} + i\zeta\mathcal{D})Y_l = (\mathcal{N} + i\zeta\mathcal{D})(u_l^+ + \sum_{j=1}^M a_j u_j^-) + O(e^{-\gamma R})$$

for $a_j = S_{lj}$. Thus Y_l gives an exponentially small discrepancy to the last equation (2.10). As approximation to the row $(S_{l1}, \dots, S_{lM})$, we take the minimizer $a^0(R) = (a_1^0(R), \dots, a_M^0(R))$ of the functional

$$(a_1, \dots, a_M) \mapsto J_l^R(a_1, \dots, a_M) = \|\mathcal{D}(\mathcal{X}_l^R - u_l^+ - \sum_{j=1}^M a_j u_j^-); L_2(\Gamma^R)\|^2, \quad (2.11)$$

where $\mathcal{X}_l^R$ is a solution to the problem (2.10). One can expect that $a_j^0(R) \to S_{lj}$ with exponential rate as $R \to \infty$ and $j = 1, \dots, M$.

To clarify the dependence of $\mathcal{X}_l^R$ on the parameters $a_1, \dots, a_M$, we consider the problems

$$\begin{aligned} (\mathcal{L}(x, D_x) - \mu)v_j^\pm &= 0, \quad x \in G^R; \\ \mathcal{B}(x, D_x)v_j^\pm &= 0, \quad x \in \partial G^R \setminus \Gamma^R; \\ (\mathcal{N} + i\zeta\mathcal{D})v_j^\pm &= (\mathcal{N} + i\zeta\mathcal{D})u_j^\pm, \ x \in \Gamma^R; \quad j = 1, \dots, M. \end{aligned} \quad (2.12)$$

It is evident that $\mathcal{X}_l^R = v_{l,R}^+ + \sum_j a_j v_{j,R}^-$, where $v_j^\pm = v_{j,R}^\pm$ are solutions to (2.12). We introduce the $(M \times M)$–matrices with entries

$$\begin{aligned} \mathcal{E}_{ij}^R &= \big(\mathcal{D}(v_i^- - u_i^-), \mathcal{D}(v_j^- - u_j^-)\big)_{\Gamma^R}, \\ \mathcal{F}_{ij}^R &= \big(\mathcal{D}(v_i^+ - u_i^+), \mathcal{D}(v_j^- - u_j^-)\big)_{\Gamma^R} \end{aligned} \quad (2.13)$$

and set

$$\mathcal{G}_i^R = \big(\mathcal{D}(v_i^+ - u_i^+), \mathcal{D}(v_i^+ - u_i^+)\big)_{\Gamma^R}.$$

Now the functional (2.11) can be written in the form

$$J_l^R(a) = \langle a\mathcal{E}^R, a\rangle + 2\mathrm{Re}\,\langle \mathcal{F}_l^R, a\rangle + \mathcal{G}_l^R,$$

where $\mathcal{F}_l^R$ is the l-th row of the matrix $\mathcal{F}^R$ and $\langle\cdot,\cdot\rangle$ is the inner product on $\mathbb{C}^M$. The minimizer a^0 (a row) satisfies $a^0(R)\mathcal{E}^R + \mathcal{F}_l^R = 0$. Therefore, as approximation $S^R(\mu)$ to the scattering matrix $S(\mu)$ we have a solution of the equation $S^R\mathcal{E}^R + \mathcal{F}^R = 0$.

Theorem 2.3. *Let ζ be any fixed number in $\mathbb{R}\setminus\{0\}$ and let the interval $[\mu_1, \mu_2]$ of the continuous spectrum of problem (2.3) be free from the threshold values of the spectral parameter μ. Then for all $\mu \in [\mu_1, \mu_2]$ and $R > R_0$, where R_0 is a sufficiently large number, there exists a unique minimizer $a^0(R, \mu) = (a_1^0(R, \mu), \dots, a_M^0(R, \mu))$ of the functional $J_l^R(\cdot;\mu)$ in (2.11). The estimates*

$$|a_j^0(R, \mu) - S_{lj}(\mu)| \leqslant Ce^{-\Lambda R}, \quad j = 1, \dots, M,$$

hold with constant C independent of μ and R, while $\Lambda = \min\{\beta, \gamma\}$, β is a positive number less than δ in (2.2), γ is the same as in (2.8).

3 Problem in the domain G^R

Introduce the boundary value problem

$$\begin{aligned} \mathcal{L}(x, D_x)u(x) - \mu u(x) &= f(x), \quad x \in G^R, \\ \mathcal{B}(x, D_x)u(x) &= g(x), \quad x \in \partial G^R \setminus \Gamma^R, \\ (\mathcal{N}(x, D_x) + i\zeta\mathcal{D}(x, D_x))u(x) &= h(x), \quad x \in \Gamma^R, \end{aligned} \tag{3.1}$$

where $\zeta \in \mathbb{R}\setminus\{0\}$ and $\mu \in \mathbb{R}$. In this section we discuss the unique solvability of the problem.

The boundary ∂G^R contains $\partial\Gamma^R$, which is an edge for $\dim G > 2$ or the union of corner points for $\dim G = 2$. The boundary conditions have discontinuities along $\partial\Gamma^R$. When studying problem (3.1), one can use the traditional scheme of the theory of elliptic boundary value problems in domains with piecewise smooth boundary (see, e.g., [5], [6], [7]). In contrast to the smooth situation, the choice of function spaces for a boundary value problem has not been universal and requires taking into account the specific properties of solutions near the edges; sometimes weighted function spaces turn out to be suitable for the purpose. For the classical problems in mathematical physics the needed spaces are known. For those reasons we restrict ourselves to considering some specific features of problem (3.1), examples, and postulating the needed properties of function spaces to be used.

Assume that the function spaces for problem (3.1) have been chosen so that the operator of the problem is Fredholm (having closed range and finite-dimensional kernel and cokernel). Then the triviality of kernel and cokernel is necessary and sufficient for the existence of a unique solution to the problem with any right-hand side. To prove the triviality, note that (2.9) leads to

$$\begin{aligned} &(\mathcal{L} - \mu)u, v)_{G^R} + (\mathcal{B}u, \mathcal{Q}v)_{\partial G^R\setminus\Gamma^R} + ((\mathcal{N} + i\zeta\mathcal{D})u, \mathcal{D}v)_{\Gamma^R} \\ &= (u, (\mathcal{L} - \mu)v)_{G^R} + (\mathcal{Q}u, \mathcal{B}v)_{\partial G^R\setminus\Gamma^R} + (\mathcal{D}u, (\mathcal{N} - i\zeta\mathcal{D})v)_{\Gamma^R}. \end{aligned} \tag{3.2}$$

Let w be a solution to the homogeneous problem (3.1). Setting $u = v = w$ in (3.2), we obtain

$$\|\mathcal{D}w; L_2(\Gamma^R)\| = 0. \tag{3.3}$$

This and the homogeneous boundary condition $(\mathcal{N}(x, D_x) + i\zeta\mathcal{D}(x, D_x))w(x) = 0$ for $x \in \Gamma^R$ imply that w has zero Cauchy data at Γ^R. If the coefficients of the operator are sufficiently smooth for the applicability of the unique continuation theorem (see [9], part II, §1.4), then we obtain the triviality of kernel. Similar considerations for the adjoint problem provide the triviality of cokernel. It is supposed that the function spaces where the problem (3.1) is considered admit such a reasoning. We illustrate this scheme of the analysis of problem (3.1) by the two following examples.

Example 1. We assume that $\dim G = 2$ and consider the problem

$$\begin{aligned}
(\Delta - \mu)u(x) &= f(x), \ \ x \in G^R, \\
u(x) &= g(x), \ \ x \in \partial G^R \setminus \Gamma^R, \\
\partial_\nu u(x) + i\zeta u(x) &= h(x), \ \ x \in \Gamma^R.
\end{aligned} \tag{3.4}$$

With every corner point of the boundary ∂G^R we associate the problem with complex parameter (operator pencil)

$$\begin{aligned}
(\partial_\omega^2 - \lambda^2)v(\omega) &= 0, \ \ \omega \in (0, \pi/2), \\
v(0) = v'(\pi/2) &= 0.
\end{aligned} \tag{3.5}$$

The spectrum of problem (3.5) consists of simple eigenvalues $\lambda_q = (2q+1)i$, where $q = 0, \pm 1, \dots$, while $\omega \mapsto \sin(2q+1)\omega$ is an eigenfunction corresponding to λ_q.

Introduce the space $V^l_\beta(G^R)$ with norm

$$\|u; V^l_\beta(G^R)\| = \left(\sum_{|\alpha| \le l} \int_{G^R} r^{2(\beta - l + |\alpha|)} |D^\alpha_x u(x)|^2 \, dx \right)^{1/2},$$

where $\beta \in \mathbb{R}$, $l = 0, 1, \dots$, and r denotes a function that coincides near a corner point with the distance to the point, equals 1 outside a neighborhood of the corner points being smooth and positive on $\overline{G^R}$ (except at the corner points). Let also $V^{l-1/2}_\beta(\partial G^R \setminus \Gamma^R)$ and $V^{l-1/2}_\beta(\Gamma^R)$ with $l = 1, 2, \dots$ stand for the space of traces of the functions in $V^l_\beta(G^R)$ on $\partial G^R \setminus \Gamma^R$ and Γ^R respectively.

The operator $\mathcal{A}^R(\mu)$ of problem (3.1) implements a continuous mapping

$$V^2_\beta(G^R) \ni u \mapsto \mathcal{A}^R(\mu)u = \{f, g, h\} \in V^0_\beta(G^R) \times V^{3/2}_\beta(\partial G^R \setminus \Gamma^R) \times V^{1/2}_\beta(\Gamma^R). \tag{3.6}$$

It is known that the operator (3.6) is Fredholm if and only if $\beta - 1$ coincides with none of the numbers $\operatorname{Im} \lambda_q$, that is, $\mathcal{A}^R(\mu)$ is Fredholm when β is not even. If w satisfies the homogeneous problem (3.1) and $w \in V^2_\beta(G^R)$ with a certain $\beta \in (2q, 2q+2)$ for an integer q, then near a corner point

$$w(x) = Cr^{2q+1} \sin(2q+1)\omega + O(r^{2q+2-\varepsilon}),$$

where r, ω are polar coordinates centered at the corner point, C is a constant, and ε is any positive number subject to $\varepsilon < 1$. Hence for every element in the kernel of the operator (3.6) with $\beta \in (0,2)$, the formula (3.3) holds, which now takes the form $\|w; L_2(\Gamma^R)\| = 0$. Therefore the kernel is trivial for $\beta \in (0,2)$ and consequently for all $\beta < 2$.

We turn to the cokernel. Denote by $V^{-l}_{-\beta}(G^R)$ the space adjoint to $V^l_\beta(G^R)$ with respect to the inner product on $L_2(G^R)$. Let $V^{-l-1/2}_{-\beta}(\Gamma^R)$ stand for the space adjoint to $V^{l+1/2}_\beta(\Gamma^R)$ with respect to the inner product on $L_2(\Gamma^R)$, $l = 0, 1$. Finally, denote by $\mathcal{A}^R(\mu)^*$ the operator adjoint to the operator (3.6),

$$\mathcal{A}^R(\mu)^* : V^0_{-\beta}(G^R) \times V^{-3/2}_{-\beta}(\partial G^R \setminus \Gamma^R) \times V^{-1/2}_{-\beta}(\Gamma^R) \to V^{-2}_{-\beta}(G^R). \tag{3.7}$$

The cokernel of operator (3.6) coincides with the kernel of operator (3.7). According to the known results on the regularity of elliptic problem solutions, for any element $\{u, v, w\}$ in the kernel of (3.7) there is the inclusion

$$\{u, v, w\} \in V^2_{2-\beta}(G^R) \times V^{1/2}_{2-\beta}(\partial G^R \setminus \Gamma^R) \times V^{3/2}_{2-\beta}(\Gamma^R), \tag{3.8}$$

the function u satisfies the homogeneous problem (3.4) with $\partial_\nu + i\zeta$ replaced for $\partial_\nu - i\zeta$ in the boundary condition at Γ^R, while v and w are determined by

$$v(x) = -\partial_\nu u(x),\ x \in \partial G^R \setminus \Gamma^R;\ \ w(x) = u(x),\ x \in \Gamma^R. \tag{3.9}$$

The above discussion of the kernel triviality of operator (3.6) does not depend on a sign of ζ. Therefore taking into account (3.8), we obtain $u = 0$ for all β such that $2 - \beta < 2$. By virtue of (3.9), for the same β we have $v = 0$ and $w = 0$. Thus if $\beta \in (0, 2)$, then both kernel and cokernel of (3.6) are trivial. It follows that for $\beta \in (0,2)$ and for all μ and $\zeta \neq 0$, the operator (3.6) is an isomorphism. Moreover, it can be shown that, for the even numbers β, the range of the operator is not closed, the cokernel is nontrivial for $\beta < 0$, and the kernel is nontrivial for $\beta > 2$. $\square$

Slightly modifying the statement of problem (3.1), it is sometimes possible to do without edges at the boundary and discontinuities in the boundary conditions. Then the analysis of the problem becomes simpler, while all the rest in the sequel requires no essential changes.

Example 2. Let the role of initial problem (2.3) be played by the Neumann problem

$$\begin{aligned} (\mathcal{L}(x, D_x) - \mu)u(x) &= f(x),\ \ x \in G, \\ \mathcal{N}(x, D_x)u(x) &= g(x),\ \ x \in \partial G, \end{aligned}$$

where $\mathcal{L}$ is the same matrix differential operator as in (2.3). Denote by $\widetilde{G}^R$ the bounded domain with smooth boundary obtained from G by cutting off the cylindrical end Π^p_+ by a surface $\widetilde{\Gamma}^{p,R}$ such that $\widetilde{\Gamma}^{p,R} \subset \{(y^p, t^p) \in \Pi^p_+ : R < t^p < R+1\}$, $p = 1, \dots, P$. As R varies, the surface $\widetilde{\Gamma}^{p,R}$ moves in a parallel way along the axis of Π^p_+. We set $\widetilde{\Gamma}^R = \cup_p \widetilde{\Gamma}^{p,R}$. Let us choose a smooth cut-off function χ on $\partial\widetilde{G}^R$ such that $\mathrm{supp}\chi \subset \widetilde{\Gamma}^R$ and $\mathrm{mes}\{x \in \widetilde{\Gamma}^R : \chi(x) = 1\} > 0$. Instead of (3.1), we introduce the problem

$$\begin{aligned} \mathcal{L}(x, D_x)u(x) - \mu u(x) &= f(x), \quad x \in \tilde{G}^R, \\ (\mathcal{N}(x, D_x) + i\zeta\chi\mathcal{D}(x, D_x))u(x) &= g(x), \quad x \in \partial\widetilde{G}^R. \end{aligned} \tag{3.10}$$

There is the Green formula

$$((\mathcal{L}-\mu)u, v)_{\tilde{G}^R} + ((\mathcal{N}+i\zeta\chi\mathcal{D})u, \mathcal{D}v)_{\partial\tilde{G}^R} = (u, (\mathcal{L}-\mu)v)_{\tilde{G}^R} + (\mathcal{D}u, (\mathcal{N}-i\zeta\chi\mathcal{D})v)_{\partial\tilde{G}^R}. \tag{3.11}$$

The operator $\mathcal{A}^R(\mu)$ of problem (3.10) implements a continuous mapping

$$\mathcal{A}^R(\mu) : \prod_{j=1}^{k} H^{l+\tau+\tau_j}(\widetilde{G}^R) \to \prod_{j=1}^{k} H^{l+\tau-\tau_j}(\widetilde{G}^R) \times \prod_{h=1}^{m} H^{l-\sigma_h-1/2}(\partial\widetilde{G}^R), \tag{3.12}$$

where $H^s(\widetilde{G}^R)$ and $H^s(\partial\widetilde{G}^R)$ are the usual Sobolev spaces, $\tau = \max\{\tau_1, \dots, \tau_k\}$, $l \geq \max\{1+\max\sigma_h, 0\}$, and the numbers k and m were defined for the matrix $\mathcal{L}(x, D_x)$ in Section 2.1; in the scalar case, $k = 1$, $\tau = \tau_1 = m$, and $\sigma_h = m_h - m$ (see 2.1). The operator (3.12) is Fredholm. From (3.11) it follows that a solution u of the homogeneous problem (3.10) satisfies $\chi(x)\mathcal{D}(x, D_x)u(x) = 0$ on $\partial\tilde{G}^R$. This and the homogeneous boundary condition in (3.10) imply that u has zero Cauchy data on the set $\{x \in \partial\tilde{G}^R : \chi(x) = 1\}$. Under the condition of unique continuation theorem, it follows that the kernel of operator (3.12) is trivial. The triviality of cokernel can be proved in a similar way.

As $\{\mathcal{L}, \mathcal{B}\}$, one can take, for instance, the operator $\{\mathcal{E}, \mathcal{N}\}$ of elasticity theory; here

$$\mathcal{E}(x, D_x)u(x) = -\mu\nabla_x \cdot \nabla_x - (\lambda+\mu)\nabla_x \cdot u(x)$$

is the Lamé system with parameters λ and μ, $\mathcal{N}(x, D_x)u(x) = \sigma(u; x)\nu(x)$, ν is the outward normal, and $\sigma(u; x)$ is the stress tensor,

$$\sigma_{jk}(u, x) = \mu\left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right) + \lambda\delta_{jk}\nabla_x \cdot u. \ \Box$$

4 Justification of the method for computing scattering matrices

To justify the method, we have to verify that the matrix $\mathcal{E}^R$ with entries (2.13) is nonsingular and the minimizer $a^0(R)$ of (2.11) tends to the l-th row of scattering matrix as $R \to \infty$.

Proposition 4.1. *Let $u_j^\pm$ be incoming and outgoing waves (2.6) satisfying (2.7). Then*

$$(\mathcal{N}u_j^\pm, \mathcal{D}u_k^\pm)_{\Gamma^R} - (\mathcal{D}u_j^\pm, \mathcal{N}u_k^\pm)_{\Gamma^R} = \mp i\delta_{jk} + O(e^{-\beta R}),$$
$$(\mathcal{N}u_j^\pm, \mathcal{D}u_k^\mp)_{\Gamma^R} - (\mathcal{D}u_j^\pm, \mathcal{N}u_k^\mp)_{\Gamma^R} = O(e^{-\beta R})$$

for $R \to +\infty$, where β is the number in the definition of $\mathcal{W}$ in 2.1, i. e., $0 < \beta < \delta$, where δ is in (2.2).

Proof. According to (2.9),

$$(\mathcal{N}u, \mathcal{D}v)_{\Gamma^R} - (\mathcal{D}u, \mathcal{N}v)_{\Gamma^R} = (u, \mathcal{L}v)_{G^R} + (\mathcal{Q}u, \mathcal{B}v)_{\partial G^R\setminus\Gamma^R} - (\mathcal{L}u, v)_{G^R} - (\mathcal{B}u, \mathcal{Q}v)_{\partial G^R\setminus\Gamma^R}.$$

If u and v are elements of the lineal $\mathcal{W}$ introduced in 2.2, then the right-hand side of the last equality differs from $q(u,v)$ by a term $O(e^{-\beta R})$ as $R \to +\infty$. Hence

$$(\mathcal{N}u_j^{\pm}, \mathcal{D}u_k^{\pm})_{\Gamma^R} - (\mathcal{D}u_j^{\pm}, \mathcal{N}u_k^{\pm})_{\Gamma^R} = q(u_j^{\pm}, u_k^{\pm}) + O(e^{-\beta R}),$$
$$(\mathcal{N}u_j^{\pm}, \mathcal{D}u_k^{\mp})_{\Gamma^R} - (\mathcal{D}u_j^{\pm}, \mathcal{N}u_k^{\mp})_{\Gamma^R} = q(u_j^{\pm}, u_k^{\mp}) + O(e^{-\beta R}).$$

It remains to take into account (2.7). $\square$

Let u be a solution to a problem of the form $\mathcal{A}^R(\mu)w = (0,0,h)$ with $(0,0,h) \in \mathcal{R}(\mathcal{A}^R(\mu) \cap L_2(\Gamma^R)$, where $\mathcal{R}(\mathcal{A}^R(\mu))$ stands for the range of the operator of problem (3.1). In what follows, we assume that $\mathcal{D}u \in L_2(\Gamma^R)$. Moreover, we also assume that if v possesses similar properties, then for u and v there holds the Green formula (2.9). (Note that such assumptions have been fulfilled for the problems in Examples 1 and 2, Section 3.)

Proposition 4.2. *The matrix $\mathcal{E}^R$ with entries (2.13) is nonsingular for all $R \geqslant R_0$, where R_0 is a sufficiently large number.*

Proof. Suppose the proposition is false. Then for any R^0 there exists a number $R > R^0$ such that the matrix $\mathcal{E}^R$ is singular, while $\mathcal{U} = \sum_j c_j u_j^-$ and $\mathcal{V} = \sum_j c_j v_j^-$ satisfy

$$\mathcal{D}\mathcal{U} = \mathcal{D}\mathcal{V} \quad \text{on } \Gamma^R, \tag{4.1}$$

where v_j^- is a solution to problem (2.12) and $\overrightarrow{c} = (c_1, \dots, c_M)$ with $|\overrightarrow{c}| = 1$. According to the equation on Γ^R in (2.12), we have

$$\mathcal{N}\mathcal{U} = \mathcal{N}\mathcal{V} \quad \text{on } \Gamma^R. \tag{4.2}$$

We set $u = v = \mathcal{V}$ in (2.9), take account of (4.1), (4.2) and of the two first equations (2.12) and obtain

$$(\mathcal{N}\mathcal{U}, \mathcal{D}\mathcal{U})_{\Gamma^R} - (\mathcal{D}\mathcal{U}, \mathcal{N}\mathcal{U})_{\Gamma^R} = 0. \tag{4.3}$$

This and Proposition 4.1 imply that

$$0 = i \sum_j |c_j|^2 + o(1) = i + o(1),$$

a contradiction. $\square$

Proposition 4.3. *Let u be a solution to the problem (3.1) with right-hand side* $(0,0,h)$, *while $h \in L_2(\Gamma^R)$. Then*

$$\|\mathcal{D}u; L_2(\Gamma^R)\| \leqslant \frac{1}{|\zeta|} \|h; L_2(\Gamma^R)\|. \tag{4.4}$$

Proof. From (2.9) it follows that

$$(\mathcal{L}u, v)_{G^R} + (\mathcal{B}u, \mathcal{Q}v)_{\partial G^R \setminus \Gamma^R} - (u, \mathcal{L}v)_{G^R} - (\mathcal{Q}u, \mathcal{B}v)_{\partial G^R \setminus \Gamma^R} =$$
$$= ((\mathcal{N} + i\zeta\mathcal{D})u, \mathcal{D}v)_{\Gamma^R} - (\mathcal{D}u, (\mathcal{N} + i\zeta\mathcal{D})v)_{\Gamma^R} - 2i\zeta(\mathcal{D}u, \mathcal{D}v)_{\Gamma^R}.$$

We set $v = u$ and obtain

$$0 = (h, \mathcal{D}u)_{\Gamma^R} - (\mathcal{D}u, h)_{\Gamma^R} - 2i\zeta\|\mathcal{D}u; L_2(\Gamma^R)\|^2.$$

Then

$$2|\zeta|\|\mathcal{D}u; L_2(\Gamma^R)\|^2 = |(h, \mathcal{D}u)_{\Gamma^R} - (\mathcal{D}u, h)_{\Gamma^R}| \leqslant 2\|\mathcal{D}u; L_2(\Gamma^R)\|\|h; L_2(\Gamma^R)\|$$

and we arrive at (4.4). $\square$

Proposition 4.4. *Let $a(R) = (a_1(R), \dots, a_M(R))$ be a minimizer of J_l^R in (2.11). Then*

$$J_l^R\,(a(R)) = O(e^{-2\gamma R}) \quad as\ R \to \infty, \tag{4.5}$$

where γ is the same as in (2.8). For all $R \geqslant R_0$,

$$|a_j(R)| \leqslant const < \infty, \quad j = 1, \dots, M.$$

Proof. Denote by Y_l^R a solution to the problem (2.10) with a_j, $j = 1, \dots, M$, equal to the entries S_{lj} of the scattering matrix S of problem (2.3). Since the asymptotics (2.8) can be differentiated, we obtain

$$(\partial_\nu + i\zeta)(Y_l^R - Y_l)|_\Gamma = O(e^{-\gamma R}).$$

The difference $Y_l^R - Y_l$ satisfies the two first equations of the problem (3.1) with $f = 0$ and $g = 0$, therefore (4.4) holds for $u = Y_l^R - Y_l$:

$$\|\mathcal{D}(Y_l^R - Y_l); L_2(\Gamma^R)\| \leqslant |\zeta|^{-1} \|(\mathcal{N} + i\zeta\mathcal{D})(Y_l^R - Y_l); L_2(\Gamma^R)\| \leqslant ce^{-\gamma R}.$$

This and (2.8) lead to the estimate

$$J_l^R(S_l) = \|\mathcal{D}(Y_l^R - (u_l^+ + \sum_{j=1}^{M} S_{lj} u_j^-)); L_2(\Gamma^R)\|^2 \leqslant ce^{-2\gamma R}$$

with constant c independent of R. Owing to $J_l^R(a(R)) \leqslant J_l^R(S_l)$, we have (4.5).

Let us estimate the minimizer $a(R)$. Denote by Z_l^R the solution of problem (2.10) corresponding to $a(R) = (a_1(R), \dots, a_M(R))$. We set $u = v = Z_l^R$ in (2.9) and obtain

$$(\mathcal{N} Z_l^R, \mathcal{D} Z_l^R)_{\Gamma^R} - (\mathcal{D} Z_l^R, \mathcal{N} Z_l^R)_{\Gamma^R} = 0. \tag{4.6}$$

By virtue of (4.5)

$$\|\mathcal{D}(Z_l^R - (u_l^+ + \sum_{j=1}^{M} a_j(R) u_j^-)); L_2(\Gamma^R)\| = O(e^{-\gamma R}), \quad R \to \infty. \tag{4.7}$$

In view of

$$(\mathcal{N} + i\zeta\mathcal{D}) Z_l^R|_{\Gamma^R} = (\mathcal{N} + i\zeta\mathcal{D})(u_l^+ + \sum_{j=1}^{M} a(R)_j u_j^-)|_{\Gamma^R},$$

from (4.7) it follows

$$\|\mathcal{N}(Z_l^R - (u_l^+ + \sum_{j=1}^{M} a_j(R) u_j^-)); L_2(\Gamma^R)\| = O(e^{-\gamma R}), \quad R \to \infty. \tag{4.8}$$

Making use of (4.7) and (4.8), we rewrite (4.6) in the form

$$(\mathcal{N}\varphi_l, \mathcal{D}\varphi_l)_{\Gamma^R} - (\mathcal{D}\varphi_l, \mathcal{N}\varphi_l)_{\Gamma^R} = O(e^{-\gamma R}),$$

where $\varphi_l = u_l^+ + \sum a_j(R)u_j^-$. According to Proposition 4.1, the left-hand side is equal to $-i(1-\sum |a_j(R)|^2) + o(1)$. Thus,

$$\sum_{j=1}^{M} |a_j(R)|^2 = 1 + o(1). \ \square$$

Proof of Theorem 2.3. Let Y_l, Z_l^R, and $(a_1(R), \dots, a_M(R))$ be the same as in Proposition 4.4. Substitute $u = v = U_l := Y_l - Z_l^R$ in the Green formula (2.9). Since U_l satisfies the two first equations in (2.10), we have

$$(\mathcal{N}U_l, \mathcal{D}U_l)_{\Gamma^R} - (\mathcal{D}U_l, \mathcal{N}U_l)_{\Gamma^R} = 0. \tag{4.9}$$

We set

$$\varphi_l = u_l^+ + \sum_{j=1}^{M} a_j(R)u_j^-, \quad \psi_l = u_l^+ + \sum_{j=1}^{M} S_{lj}u_j^- \tag{4.10}$$

and rewrite U_l in the form

$$U_l - Y_l - Z_l^R = (Y_l - \psi_l) + (\psi_l - \varphi_l) + (\varphi_l - Z_l^R).$$

Note that $(Y_l - \psi_l)|_{\Gamma^R} = O(e^{-\gamma R})$ by virtue of (2.8). In view of (4.7), (4.8), and Proposition 4.4, this enables us to pass from (4.9) to

$$(\mathcal{N}(\psi_l - \varphi_l), \mathcal{D}(\psi_l - \varphi_l))_{\Gamma^R} - (\mathcal{D}(\psi_l - \varphi_l), \mathcal{N}(\psi_l - \varphi_l))_{\Gamma^R} = O(e^{-\Lambda R}), \tag{4.11}$$

where $\Lambda = \min\{\beta, \gamma\}$. The left-hand side can be immediately calculated and is equal to

$$i\sum_{j=1}^{M} |a_j(R) - S_{lj}|^2 + O(e^{-\beta R});$$

to see that, it suffices to use (4.10) and Proposition 4.1. Finally, we obtain

$$\sum_{j=1}^{M} |a_j(R) - S_{lj}|^2 = O(e^{-\Lambda R}). \ \square$$

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